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S-Dominating Effect Algebras

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A special type of effect algebra called an S-dominating effect algebra is introduced. It is shown that an S-dominating effect algebra P has a naturally defined Brouwer-complementation that gives P the structure of a Brouwer–Zadeh poset. This enables us to prove that the sharp elements of P form an orthomodular lattice. We then show that a standard Hilbert space effect algebra is S-dominating. We conclude that S-dominating effect algebras may be useful abstract models for sets of quantum effects in physical systems.

1. INTRODUCTION

Effect algebras (or D-posets) have recently been introduced for investigating the foundations of quantum mechanics (Dvurečenskij, 1995; Dvurečenskij and Pulmannová, 1994; Foulis and Bennett, 1994; Kôpka, 1992; Kôpka and Chovanec, 1994; Riečanová and Brsel, 1994). The advantage of effect algebras over previously defined structures of sharp elements such as orthoalgebras (Feldman and Wilce, 1933; Foulis *et al.*, 1992; Gudder, 1988) and orthomodular posets (Beltrametti and Cassinelli, 1981; Gudder, 1988; Pták and Pulmannová, 1991) is that effect algebras provide a mechanism for studying quantum effects that may be unsharp. However, an effect algebra is so general that its set of sharp elements need not form a regular algebraic structure. To remedy this shortcoming, we introduce a special type of effect algebra called S-dominating. We show that for S-dominating effect algebras, the set of sharp elements form an orthomodular lattice. We also show that a standard Hilbert space effect algebra is S-dominating. Hence, S-dominating effect algebras may be useful abstract models for sets of quantum effects in physical systems.

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We now give a brief overview, leaving precise mathematical definitions for Section 2. We first consider a generalization of an effect algebra called a DeMorgan (DM) poset P. We denote the sharp elements of P by P_s and say that P is sharply dominating if every element of P is dominated by a smallest sharp element. If P is sharply dominating, it is shown that P possesses a natural B-complementation that gives P the structure of a BZ-poset. For BZ-posets, we show that existing infima and suprema of sharp elements are sharp. A sharply dominating DM-poset P is called an S-dominating DM-poset if $a \wedge p$ exists for every $a \in P$, $p \in P_s$. It follows that if P is S-dominating, then P_s is an orthocomplemented lattice. We next consider S-dominating effect algebras and show in this case that P_s is an orthomodular lattice. Finally, Section 3 shows that Hilbert space effect algebras are S-dominating.

2. S-DOMINATING STRUCTURES

A DM-*poset* is an algebraic structure $(P, \leq, 0, 1, ')$ where $(P, \leq, 0, 1)$ is a bounded poset and $'$ is a unary operation on P that satisfies: $a'' = a$ and $a \leq b$ implies $b' \leq a'$. It is easy to verify that DeMorgan's laws hold on P. That is, $(a \vee b)' = a' \wedge b'$ and $(a \wedge b)' = a' \vee b'$ in the sense that if one side of the equality exists, then so does the other side and they coincide. An element $a \in P$ is *sharp* if $a \wedge a'$ exists and equals 0. It is clear that 0, 1 are sharp and if a is sharp, then a' is sharp. Denoting the set of sharp elements in P by P_s, it follows that $(P_s, \leq, 0, 1, ')$ is an orthocomplemented poset. We say that P is *sharply dominating* if every $a \in P$ is dominated by a smallest sharp element $\hat{a}$. That is, $\hat{a} \in P_s$, $a \leq \hat{a}$ and if $b \in P_s$ satisfies $a \leq b$, then $\hat{a} \leq b$. It is evident that $\hat{a}$ is unique. If P is a sharply dominating DM-poset, we define a unary operation $\sim$ on P by $a^\sim = (\hat{a})'$. The following result is proved in Gudder (n.d.).

Lemma 2.1. If P is a sharply dominating DM-poset, then for every a, $b \in P$ we have (i) $a \leq a^{\sim\sim}$, (ii) $a \leq b$ implies $b^\sim \leq a^\sim$, (iii) $a \wedge a^\sim = 0$, (iv) $a^{\sim\prime} = a^{\sim\sim}$.

A unary operation $\sim$ that satisfies (i)–(iv) of Lemma 2.1 is called a B-*complementation* and a DM-poset with a B-complementation $(P, \leq, 0, 1, ', \sim)$ is called a BZ-poset (Cattaneo, n.d.; Cattaneo and Marino, 1988; Cattaneo and Nisticò, 1989; Gudder, n.d.). The results in the next lemma are proved in Gudder (1996, n.d.).

Lemma 2.2. Let P be a BZ-poset. (i) If $a \vee b$ exists in P, then $a^\sim \wedge b^\sim$ exists in P and $(a \vee b)^\sim = a^\sim \wedge b^\sim$. (ii) The following statements are equivalent: (1) $a \in P_s$, (2) $a = a^{\sim\sim}$, (3) $a' = a^\sim$.

Corollary 2.3. Let P be a BZ-poset and let $a, b \in P_s$. (i) If $a \vee b$ exists, then $a \vee b \in P_s$. (ii) If $a \wedge b$ exists, then $a \wedge b \in P_s$.

Proof. (i) Applying Lemma 2.2, we have

$$(a \vee b)^{\sim} = a^{\sim} \wedge b^{\sim} = a' \wedge b' = (a \vee b)'$$

Hence, by Lemma 2.2(ii), we have that $a \vee b \in P_s$.

(ii) Since $a, b \in P_s$, we have $a', b' \in P_s$. By (i), we have $(a \wedge b)' = a' \vee b' \in P_s$. ■

Of course, if P is a sharply dominating DM-poset, then P is a BZ-poset, so Lemma 2.2 and Corollary 2.3 hold for P. A sharply dominating DM-poset P is called a S-*dominating* DM-*poset* if $a \wedge p$ exists for every $a \in P$, $p \in P_s$. (It follows from DeMorgan's laws that $a \vee p$ also exists.) If P is an S-dominating DM-poset, it follows from Corollary 2.3 that $(P_s, \leq, 0, 1, ')$ is an orthocomplemented lattice.

Lemma 2.4. Let P be an S-dominating DM-poset. (i) If $a, b \in P$, $p_1, p_2 \in P_s$, and $a \wedge b$ exists, then $(a \wedge p_1) \wedge (b \wedge p_2)$ exists and equals $(a \wedge b) \wedge (p_1 \wedge p_2)$. (ii) If $a, b \in P$, $p \in P_s$, and $a \wedge b$ exists, then $(a \wedge p) \wedge b$ exists and equals $(a \wedge b) \wedge p$.

Proof. (i) Notice that if $a \wedge b$ exists, then $(a \wedge b) \wedge (p_1 \wedge p_2)$ automatically exists. Now

$$(a \wedge b) \wedge (p_1 \wedge p_2) \leq a \wedge p_1, \quad b \wedge p_2$$

and suppose that $c \leq a \wedge p_1, b \wedge p_2$. Then $c \leq a, b$, so $c \leq a \wedge b$. Since $c \leq p_1, p_2$ we also have $c \leq p_1 \wedge p_2$. Hence, $c \leq (a \wedge b) \wedge (p_1 \wedge p_2)$ and the result follows. (ii) This follows from (i) with $p_2 = 1$. ■

Lemma 2.4(i) states that the "global" existence of $a \wedge b$ implies the "local" existence $(a \wedge p_1) \wedge (b \wedge p_2)$. Letting $p_1 = p_2 = 0$ shows that the converse of Lemma 2.4(i) does not hold. For $a \in P$, the element $\hat{a} \in P_s$ corresponds to the "support" of a in a certain sense. (This idea will become clearer in Section 3.) Moreover, for $a, b \in P$, the element $p_{a,b} = \hat{a} \wedge \hat{b} \in P_s$ corresponds to the "common support" of a and b. The next result shows that $a \wedge b$ exists if and only if their infimum exists on their common support. For a discussion of the existence of $a \wedge b$ in the Hilbert space context, see Moreland and Gudder (n.d.).

Theorem 2.5. Let P be an S-dominating DM-poset and let $a, b \in P$. (i) $a \wedge b$ exists if and only if $(a \wedge \hat{b}) \wedge (b \wedge \hat{a})$ exists and in this case they coincide. (ii) $a \wedge b$ exists if and only if $(a \wedge p_{a,b}) \wedge (b \wedge p_{a,b})$ exists and in this case they coincide.

Proof. (i) If $a \wedge b$ exists, then by Lemma 2.4(i), $(a \wedge \hat{b}) \wedge (b \wedge \hat{a})$ exists and

$$(a \wedge \hat{b}) \wedge (b \wedge \hat{a}) = (a \wedge b) \wedge (\hat{a} \wedge \hat{b})$$

Since $a \wedge b \leq a \leq \hat{a}$ and $a \wedge b \leq b \leq \hat{b}$, we have $a \wedge b \leq \hat{a} \wedge \hat{b}$. Hence, $(a \wedge b) \wedge (\hat{a} \wedge \hat{b}) = a \wedge b$. Conversely, suppose that $(a \wedge \hat{b}) \wedge (b \wedge \hat{a})$ exists. Then $(a \wedge \hat{b}) \wedge (b \wedge \hat{a}) \leq a, b$. If $c \leq a, b$, then $c \leq \hat{a}, \hat{b}$, so $c \leq a \wedge \hat{b}, b \wedge \hat{a}$. Hence, $c \leq (a \wedge \hat{b}) \wedge (b \wedge \hat{a})$ and the result follows. (ii) Applying Lemma 2.4(ii), we have

$$a \wedge p_{a,b} = (\hat{a} \wedge \hat{b}) \wedge a = (\hat{a} \wedge a) \wedge \hat{b} = a \wedge \hat{b}$$

and similarly, $b \wedge p_{a,b} = b \wedge \hat{a}$. The result now follows from (i). ■

An *effect algebra* is an algebraic structure $(P, \oplus, 0, 1)$ where 0, 1 are distinct elements of P and $\oplus$ is a partial binary operation on P that satisfies the following conditions:

(E1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $b \oplus a = a \oplus b$.
(E2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
(E3) For every $a \in P$, there exists a unique $a' \in P$ such that $a \oplus a' = 1$.
(E4) If $a \oplus 1$ is defined, then $a = 0$.

If $a \oplus b$ is defined, we write $a \perp b$ and whenever we write $a \oplus b$ we are implicitly assuming that $a \perp b$. We define $a \leq b$ if there exists a $c \in P$ such that $a \oplus c = b$. It can be shown that $a \perp b$ if and only if $a \leq b'$. Moreover, $(P, \leq, 0, 1, ')$ forms a DM-poset. An *orthoalgebra* is an algebraic structure $(P, \oplus, 0, 1)$ that satisfies (E1)–(E3) and the following condition:

(E5) If $a \oplus a$ is defined, then $a = 0$.

It is easy to show that an orthoalgebra is a special case of an effect algebra.

Since an effect algebra P is a DM-poset, our previous definitions carry over to P. Thus, $a \in P$ is *sharp* if $a \wedge a' = 0$. Also, P is *sharply dominating* or S-*dominating* if P has these properties as a DM-poset. In any effect algebra the following *effect algebra orthomodular identity* holds:

$$a \leq b \quad \text{implies} \quad a \oplus (a \oplus b')' = b$$

Indeed, since

$$b' \oplus a \oplus (a \oplus b')' = 1 = b' \oplus b$$

the identity follows by cancellation (Foulis and Bennett, 1994). An *orthomodular lattice* is an orthocomplemented lattice in which the following *lattice orthomodular identity* holds:

$$a \leq b \quad \text{implies} \quad a \vee (b \wedge a') = b$$

It is shown in Cattaneo (n.d.) and Gudder (n.d.) that for a sharply dominating effect algebra P, the set P_s is an orthoalgebra. The following theorem shows that if P is an S-dominating effect algebra, then P_s is an orthomodular lattice.

Theorem 2.6. Let P be an S-dominating effect algebra and let $a \in P$, $p \in P_s$. (i) If $a \perp p$, then $a \vee p = a \oplus p$. (ii) If $a' \perp p'$, then $a \wedge p = (a' \oplus p')'$. (iii) If $a \leq p$, then $a \oplus (p \wedge a') = p$. (iv) If $p \leq a$, then $p \vee (a \wedge p') = a$. (v) P_s is an orthomodular lattice.

Proof. (i) It is shown in Cattaneo (n.d.) and Gudder (n.d.) that in any effect algebra, if $a \perp p$, then $a \oplus p$ is a minimal upper bound for a and b. Since $a \vee p$ exists, it follows that $a \vee p = a \oplus p$. (ii) Applying (i), we have

$$a \wedge p = (a' \vee p')' = (a' \oplus p')'$$

(iii) By the effect algebra orthomodular identity and (i) we have

$$p = a \oplus (a \oplus p')' = a \oplus (a \vee p')' = a \oplus (p \wedge a')$$

(iv) As in (iii) we have

$$a = p \oplus (p \oplus a')' = p \oplus (a' \vee p)' = p \oplus (a \wedge p') = p \vee (a \wedge p')$$

(v) We have already noted that $(P_s, \leq, 0, 1, ')$ is an orthocomplemented lattice. If $p, q \in P_s$ with $p \leq q$, then applying (iv), we have $p \vee (q \wedge p') = q$. Hence, the lattice orthomodular identity holds. ■

3. HILBERT SPACE EFFECT ALGEBRAS

The most important example of an effect algebra for quantum mechanical investigations is a Hilbert space effect algebra. Let H be a complex Hilbert space and let $\mathscr{E}(H)$ be the set of linear operators on H that satisfy $0 \leq A \leq I$. That is, $0 \leq \langle Ax, x\rangle \leq \langle x, x\rangle$ for all $A \in \mathscr{E}(H)$ and $x \in H$. For $A, B \in \mathscr{E}(H)$ we write $A \perp B$ if $A + B \in \mathscr{E}(H)$ and in this case we define $A \oplus B = A + B$. If we define $A' = I - A$ for $A \in \mathscr{E}(H)$, it is clear that $(\mathscr{E}(H), \oplus, 0, I)$ is an effect algebra which we call a *Hilbert space effect algebra*. Denoting the set of projections on H by $\mathscr{P}(H)$, we have $\mathscr{P}(H) \subseteq \mathscr{E}(H)$ and it can be shown that $\mathscr{P}(H)$ is an orthomodular lattice. (This result will also follow from Theorem 2.6.) Moreover, it can be shown that $\mathscr{E}(H)_s = \mathscr{P}(H)$ (Gudder, 1996).

We now show that $\mathscr{E}(H)$ is S-dominating. Since $\mathscr{E}(H)$ is the standard concrete model for the quantum effects of a physical system (Busch *et al.*, 1991; Davies, 1976; Holevo, 1982; Kraus, 1983; Ludwig, 1983), this result shows that an S-dominating effect algebra gives a viable abstract model for quantum mechanics. We first verify that $\mathscr{E}(H)$ is sharply dominating. For $A \in \mathscr{E}(H)$, let $\hat{A}$ be the projection onto the closure of the range $R(A)$ of A. Then $\hat{A} \in \mathscr{P}(H) = \mathscr{E}(H)_s$ and since $A\hat{A} = \hat{A}A = A$, it follows that $A \leq \hat{A}$. If $B \in \mathscr{P}(H)$ and $A \leq B$, then the null space $N(B) \subseteq N(A)$. Hence,

$$\overline{R(A)} = N(A)^{\perp} \subseteq N(B)^{\perp} = \overline{R(B)}$$

It follows that $\hat{A} \leq B$. Hence, $\hat{A}$ is the smallest sharp element that dominates A.

We now show that if $A \in \mathscr{E}(H)$ and $P \in \mathscr{P}(H)$, then $A \wedge P$ exists. Our demonstration is similar to the proof given in Moreland and Gudder (n.d.), where a more general problem is considered. We include the proof here for completeness and because of its independent interest. The next lemma is a well-known result.

Lemma 3.1. If A is a positive operator on H, then

$$\left|\langle Ax, y\rangle\right|^2 \leq \langle Ax, x\rangle\langle Ay, y\rangle$$

for every $x, y \in H$.

Proof. Since $A \geq 0$, A admits a unique positive square root $A^{1/2}$. By Schwarz's inequality, we have

$$\begin{aligned}\left|\langle Ax, y\rangle\right|^2 &= \left|\langle A^{1/2}x, A^{1/2}y\rangle\right|^2 \leq \left\|A^{1/2}x\right\|^2\left\|A^{1/2}y\right\|^2\\ &= \langle A^{1/2}x, A^{1/2}x\rangle\langle A^{1/2}y, A^{1/2}y\rangle = \langle Ax, x\rangle\langle Ay, y\rangle \quad \blacksquare\end{aligned}$$

Lemma 3.2. Let $e_1, e_2, \ldots$ be an orthonormal set in H and let $P_n \in \mathscr{P}(H)$ be the projection onto the subspace spanned by $\{e_1, \ldots, e_n\}$. If $A \in \mathscr{E}(H)$, then $A \wedge P_n'$ exists.

Proof. We first show that $A \wedge P_1'$ exists. Let $a = \langle Ae_1, e_1\rangle$. If $a = 0$, then

$$\left\|A^{1/2}e_1\right\|^2 = \langle A^{1/2}e_1, A^{1/2}e_1\rangle = a = 0$$

Hence, $A^{1/2}e_1 = 0$, so $Ae_1 = 0$. It follows that $A = P_1'AP_1' \leq P_1'$. Hence, $A \wedge P_1' = A$. Now suppose that $a > 0$ and define the operator $B = a^{-1}AP_1A$. It is evident that $B \geq 0$. Applying Lemma 3.1, we have

$$\begin{aligned}\langle Bx, x\rangle &= a^{-1}\langle AP_1Ax, x\rangle = a^{-1}\langle P_1Ax, Ax\rangle = a^{-1}\left|\langle Ax, e_1\rangle\right|^2\\ &\leq a^{-1}\langle Ae_1, e_1\rangle\langle Ax, x\rangle = \langle Ax, x\rangle\end{aligned}$$

Hence, $B \leq A$, so $C = A - B \geq 0$. Moreover, $C \leq A \leq I$ and $C \in \mathscr{E}(H)$. Since

$$Be_1 = a^{-1}AP_1Ae_1 = a^{-1}A(\langle Ae_1, e_1\rangle e_1) = Ae_1$$

we have $Ce_1 = 0$. Hence, $C = P_1'CP_1' \leq P_1'$. To show that $C = A \wedge P_1'$, suppose that $D \in \mathscr{E}(H)$ and $D \leq A, P_1'$. Then $D = P_1'DP_1'$ and $De_1 = 0$. Applying Lemma 3.1, we have

$$\begin{aligned}\langle P_1' B P_1' x, x\rangle &= a^{-1}\langle P_1' A P_1 A P_1' x, x\rangle = a^{-1}\langle P_1' A(\langle A P_1' x, e_1\rangle e_1), x\rangle \\ &= a^{-1}\langle A P_1' x, e_1\rangle\langle P_1' A e_1, x\rangle = a^{-1}\left|\langle A e_1, P_1' x\rangle\right|^2 \\ &= a^{-1}\left|\langle (A - D)e_1, P_1' x\rangle\right|^2 \\ &\le a^{-1}\langle (A - D)e_1, e_1\rangle\langle (A - D)P_1' x, P_1' x\rangle \\ &= \langle P_1'(A - D)P_1' x, x\rangle\end{aligned}$$

Hence, $P_1' B P_1' \le P_1'(A - D)P_1'$. We then have

$$C - D = P_1'(C - D)P_1' = P_1'(A - D)P_1' - P_1' B P_1' \ge 0$$

Thus, $D \le C$ and we conclude that $C = A \wedge P_1'$.

We next show that $A \wedge P_2'$ exists. Since $A \wedge P_1' \le P_1'$, we can identify $A \wedge P_1'$ with the restriction $A \wedge P_1' \big| P_1' H$. Proceeding as before, we conclude that $(A \wedge P_1') \wedge P_2'$ exists. It is clear that $(A \wedge P_1') \wedge P_2' \le A, P_2'$. Suppose that $D \in \mathcal{E}(H)$ satisfies $D \le A, P_2'$. Since $P_2' \le P_1'$, we have $D \le A \wedge P_1', P_2'$, so $D \le (A \wedge P_1') \wedge P_2'$. Hence,

$$A \wedge P_2' = (A \wedge P_1') \wedge P_2'$$

Continuing by induction, we conclude that $A \wedge P_n'$ exists for all $n \in \mathbb{N}$. ■

Theorem 3.3. If $A \in \mathcal{E}(H)$ and $P \in \mathcal{P}(H)$, then $A \wedge P$ exists.

Proof. If P' is finite-dimensional, we are finished, by Lemma 3.2, so suppose P' is infinite-dimensional. Let $\{f_\delta : \delta \in \Delta\}$ be an orthonormal basis for $P'H$. For $\alpha \subseteq \Delta$ with cardinality $|\alpha| < \infty$, let Q_α be the projection onto the closed subspace spanned by $\{f_\delta : \delta \notin \alpha\}$. Then $\{\alpha \subseteq \Delta : |\alpha| < \infty\}$ is a directed set under set-theoretic inclusion and $\{Q_\alpha : \alpha \subseteq \Delta, |\alpha| < \infty\}$ is a decreasing net of projections. Moreover, $(P + Q_\alpha)'$ is the projection onto the finite-dimensional subspace spanned by $\{f_\delta : \delta \in \alpha\}$. For $x \in H$, since

$$\sum \left\{\left|\langle x, f_\delta\rangle\right|^2 : \delta \in \Delta\right\} < \infty$$

we have

$$\lim\langle Q_\alpha x, x\rangle = \lim \sum \left\{\left|\langle x, f_\delta\rangle\right|^2 : \delta \notin \alpha\right\} = 0$$

Hence,

$$\lim\langle (P + Q_\alpha)x, x\rangle = \langle Px, x\rangle$$

for every $x \in H$. Since $(P + Q_\alpha)'$ is finite-dimensional, by Lemma 3.2, $A \wedge (P + Q_\alpha)$ exists. Since $A \wedge (P + Q_\alpha)$ is a decreasing net of positive operators, it follows from a well-known theorem (Brown and Page, 1970) that there exists a $B \in \mathscr{E}(H)$ such that

$$\langle Bx, x\rangle = \lim\langle A \wedge (P + Q_\alpha)x, x\rangle$$

for every $x \in H$. Since $A \wedge (P + Q_\alpha) \leq A$, we have $\langle Bx, x\rangle \leq \langle Ax, x\rangle$ for every $x \in H$, so $B \leq A$. Since $A \wedge (P + Q_\alpha) \leq P + Q_\alpha$, we have

$$\langle Bx, x\rangle \leq \lim\langle (P + Q_\alpha)x, x\rangle = \langle Px, x\rangle$$

for every $x \in H$, so $B \leq P$. Suppose that $C \in \mathscr{E}(H)$ and $C \leq A, P$. Then $C \leq A, P + Q_\alpha$, so $C \leq A \wedge (P + Q_\alpha)$ for every α. Hence, for every $x \in H$, we have

$$\langle Cx, x\rangle \leq \lim\langle A \wedge (P + Q_\alpha)x, x\rangle = \langle Bx, x\rangle$$

Therefore, $C \leq B$, so $B = A \wedge P$. ∎

Applying Theorems 2.6 and 3.3, we can draw some interesting conclusions. If $A \in \mathscr{E}(H)$, $P \in \mathscr{P}(H)$, and $A + P \leq I$, then $A \vee P = A + P$. If $A \in \mathscr{E}(H)$, $P \in \mathscr{P}(H)$, and $I \leq A + P$, then $A \wedge P = A + P - I$. This last property can be restated as follows. If $A' \leq P$, then $A \wedge P = P - A' = A - P'$.

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